Variational Methods

consider a lossless inhomogeneous region enclosed by a perfect conductor, and say we were interested in the resonant Frequency, then in the region:

$$\bar\nabla\times\bar E = -j\omega_r\mu\bar H \qquad\qquad \bar\nabla\times\bar H = j\omega_r\epsilon\bar E$$

$$\mu^{-1}\bar\nabla\times\bar E = -j\omega_r\bar H \qquad\qquad \epsilon^{-1}\bar\nabla\times\bar H = j\omega_r\bar E$$

$$\bar\nabla\times\mu^{-1}\bar\nabla\times\bar E = -j\omega_r\bar\nabla\times\bar H \qquad\qquad \bar\nabla\times\epsilon^{-1}\bar\nabla\times\bar H = j\omega_r\bar\nabla\times\bar E$$

$$\boxed{\bar\nabla\times\mu^{-1}\bar\nabla\times\bar E - \omega_r^2\epsilon\bar E = 0} \qquad \boxed{\bar\nabla\times\epsilon^{-1}\bar\nabla\times\bar H - \omega_r^2\mu\bar H = 0}$$

multiplying the First by $\bar E\cdot$ and integrating throughout the enclosed region:

$$\boxed{\omega_r^2 = \frac{\iiint \bar E\cdot\bar\nabla\times\mu^{-1}\bar\nabla\times\bar E\,dv}{\iiint \epsilon E^2\,dv}}$$

$$\left(E^2 = E_x^2 + E_y^2 + E_z^2\right)$$

also
$$\boxed{\omega_r^2 = \frac{\iiint \bar H\cdot\bar\nabla\times\epsilon^{-1}\bar\nabla\times\bar H\,dv}{\iiint \mu H^2\,dv}}$$

the above formulas for ω_r^2 are "stationary":

consider a trial field:

$$\bar{E}_{trial} = \bar{E} + \Delta\bar{E} = \bar{E} + p\bar{e}$$

where $\bar{E}$ is the true electric field and p is an arbitrary parameter which we use to vary the trial solution.

$$\therefore \quad \omega^2(p) = \frac{\iiint (\bar{E}+p\bar{e})\cdot\bar{\nabla}\times\mu^{-1}\bar{\nabla}\times(\bar{E}+p\bar{e})\,dv}{\iiint \epsilon(\bar{E}+p\bar{e})\cdot(\bar{E}+p\bar{e})\,dv}$$

notice $\omega^2(0) = \omega_r^2$

expanding $\omega^2(p)$ in a <u>Maclaurin series</u>:

$$\omega^2(p) = \omega_r^2 + p\left.\frac{\partial\omega^2}{\partial p}\right|_{p=0} + \frac{p^2}{2!}\left.\frac{\partial^2\omega^2}{\partial p^2}\right|_{p=0} + \cdots$$

now it is standard procedure to define

$$\left.\frac{\partial\omega^2}{\partial p}\right|_{p=0} \triangleq \delta\omega^2 \qquad \text{(first variation)}$$

$$\left.\frac{\partial^2\omega^2}{\partial p^2}\right|_{p=0} \triangleq \delta^2\omega^2 \qquad \text{(second variation)}$$

$$\therefore \quad \boxed{\omega^2(p) = \omega_r^2 + p\,\delta\omega^2 + \frac{p^2}{2!}\delta^2\omega^2 + \dots}$$

if the First variation vanishes ($\delta\omega^2 = 0$) then the formula is said to be stationary.

We can show that our formulas are stationary:

let $N(p) = \iiint (\bar{E}+p\bar{e})\cdot\bar{\nabla}\times\mu^{-1}\bar{\nabla}\times(\bar{E}+p\bar{e})\,dv$

$$N'(p) = \iiint \begin{matrix} \bar{e}\cdot\bar{\nabla}\times\mu^{-1}\bar{\nabla}\times(\bar{E}+p\bar{e}) \\ +(\bar{E}+p\bar{e})\cdot\bar{\nabla}\times\mu^{-1}\bar{\nabla}\times\bar{e} \end{matrix}\, dv$$

$$N'(0) = \iiint \bar{E}\cdot\bar{\nabla}\times\mu^{-1}\bar{\nabla}\times\bar{e} + \bar{e}\cdot\bar{\nabla}\times\mu^{-1}\bar{\nabla}\times\bar{E}\,dv$$

by an identity:

$$\iiint \bar{E}\cdot\bar{\nabla}\times\mu^{-1}\bar{\nabla}\times\bar{e}\,dv \equiv \iiint \mu^{-1}\bar{\nabla}\times\bar{e}\cdot\bar{\nabla}\times\bar{E}\,dv + \oiint [(\mu^{-1}\bar{\nabla}\times\bar{e})\times\bar{E}]\cdot\hat{n}\,ds$$

now since $((\mu^{-1}\bar{\nabla}\times\bar{e})\times\bar{E})\cdot\hat{n} = (\bar{E}\times\hat{n})\cdot\mu^{-1}\bar{\nabla}\times\bar{e}$
and $\bar{E}\times\hat{n} = 0$ due to B.C's.
the last term vanishes.

$$\left(\begin{aligned} N(0) &= \iiint \bar{E}\cdot\bar{\nabla}\times\mu^{-1}\bar{\nabla}\times\bar{E}\,dv \\ &= \omega_r^2 \iiint \varepsilon E^2\,dv \end{aligned} \right)$$

similarly:

(same eq.) $\bar{E} \Leftrightarrow \bar{e}$

$$\iiint \mu^{-1} \bar{\nabla}\times\bar{e} \cdot \bar{\nabla}\times\bar{E} \, dv = \iiint \bar{e}\cdot\bar{\nabla}\times\mu^{-1}\bar{\nabla}\times\bar{E}\, dv - \oiint \left[(\mu^{-1}\bar{\nabla}\times\bar{E})\times\bar{e}\right]\cdot d\bar{s}$$

$$\therefore \quad N'(0) = \iiint \mu^{-1}\bar{\nabla}\times\bar{e}\cdot\bar{\nabla}\times\bar{E}\, dv + \iiint \bar{e}\cdot\bar{\nabla}\times\mu^{-1}\bar{\nabla}\times\bar{E}\, dv$$

$\bar{\nabla}\times\mu^{-1}\bar{\nabla}\times\bar{E} = \omega_r^2\varepsilon\bar{E}$

$$= 2\iiint \bar{e}\cdot\bar{\nabla}\times\mu^{-1}\bar{\nabla}\times\bar{E}\, dv - \oiint \left[(\mu^{-1}\bar{\nabla}\times\bar{E})\times\bar{e}\right]\cdot d\bar{s}$$

$$N'(0) = 2\omega_r^2 \iiint \epsilon\, \bar{e}\cdot\bar{E}\, dv - \oiint\left[(\mu^{-1}\bar{\nabla}\times\bar{E})\times\bar{e}\right]\cdot d\bar{s}$$

let the denominator $D(p) = \iiint \epsilon (\bar{E}+p\bar{e})\cdot(\bar{E}+p\bar{e})\, dv$

$$\therefore \quad D'(0) = 2\iiint \epsilon\, \bar{e}\cdot\bar{E}\, dv$$

$$\therefore \quad \delta\omega^2 = \left.\frac{\partial\omega^2}{\partial p}\right|_{p=0} = \frac{D(0)N'(0) - N(0)D'(0)}{D^2(0)} \quad \text{(quotient rule).}$$

$$= \frac{\iiint \epsilon E^2 dv \; N'(0) - \iiint \omega_r^2 \epsilon E^2 dv \; 2\iiint \epsilon\, \bar{e}\cdot\bar{E}\, dv}{\left[\iiint \epsilon E^2 dv\right]^2}$$

$$\boxed{\delta\omega^2 = -\frac{\oiint [(\mu^{-1}\bar{\nabla}\times\bar{E})\times\bar{e}]\cdot d\bar{s}}{\iiint \epsilon E^2 dv}}$$

if $\hat{n}\times\bar{e}=0$ on S then $\delta\omega^2=0$

this implies that $\hat{n}\times\bar{E}_{trial}=0$ on S.

Thus the formula for $\omega^2(p)$ is stationary if the trial $\bar{E}$ satisfy the proper B.C's.

An <u>alternative</u> formula for ω_r^2 can be found as:

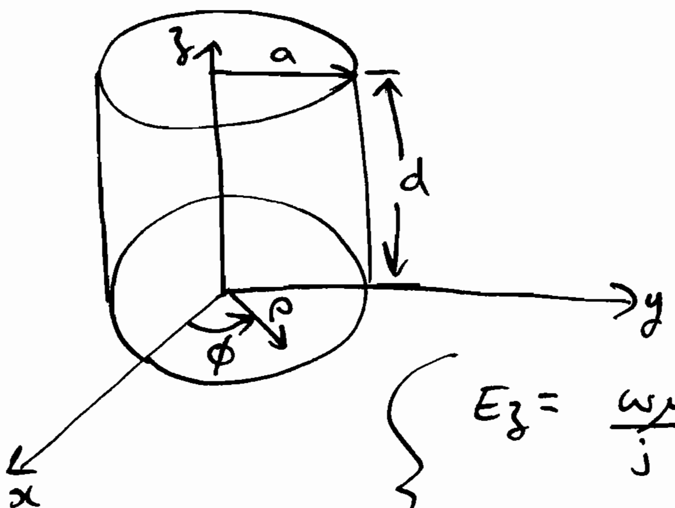
$$\omega_r^2 = \frac{\iiint \mu^{-1}(\bar{\nabla}\times\bar{E})^2 dv}{\iiint \epsilon E^2 dv}$$

and is stationary if

$\hat{n}\times\bar{E}_{trial}=0$ or if $2\oiint[(\mu^{-1}\bar{\nabla}\times\bar{E})\times\bar{E}]\cdot d\bar{s}$ is added to the Numerator

Stationary formulas give a smaller error in ω_r^2 w.r.t. the amount of error in trial solutions than non-stationary formulas.

<u>Example</u>: The TM_{010} mode is dominant for a cylindrical cavity where $d < 2a$



$$\psi_{npq}^{TM} = J_n\left(\frac{x_{np}\,\rho}{a}\right)\begin{Bmatrix}\sin n\phi \\ \cos n\phi\end{Bmatrix}\cos\left(\frac{q\pi}{d}z\right)$$

$$\psi_{010}^{TM} = J_0\left(\frac{x_{01}\,\rho}{a}\right) = J_0\left(\frac{2.405}{a}\rho\right)$$

$$\begin{cases} E_z = \dfrac{\omega\mu}{j} J_0\left(2.405\dfrac{\rho}{a}\right) \\ \\ H_\phi = \dfrac{2.405}{a} J_1\left(2.405\dfrac{\rho}{a}\right) \end{cases}$$

$$\omega_r = \frac{2.405}{a\sqrt{\mu\epsilon}} \qquad \left(k^2 = 0^2 + 0^2 + \frac{x_{01}^2}{a^2} = \frac{\omega^2}{c^2} \right)$$

(separation eq.)

now consider the stationary formula

$$\omega_r^2 = \frac{\iiint \mu^{-1}(\bar{\nabla}\times\bar{E})^2\,dv}{\iiint \epsilon E^2\,dv} \qquad \left(\text{require } \hat{n}\times\bar{E}_{trial} = 0\right)$$

choose : $\bar{E} = \hat{a}_z\left(1 - \frac{\rho}{a}\right) \Rightarrow \bar{\nabla}\times\bar{E} = \hat{a}_\phi \frac{1}{a}$

$$\therefore\ \omega^2 = \frac{\int_0^a \frac{\rho}{a^2}\,d\rho}{\mu\epsilon \int_0^a \left(1-\frac{\rho}{a}\right)^2 \rho\,d\rho} = \frac{6}{\epsilon\mu a^2} \Rightarrow \omega = \frac{2.449}{a\sqrt{\mu\epsilon}}$$

1.8% <u>error</u>

Rayleigh- Ritz Procedure

we define the trial function in terms of n variational parameters $(A_1, \ldots A_n)$

i.e. $\bar{E}_{trial} = \bar{E}_{trial}(A_1, \ldots A_n)$

substituting this trial fcn into the stationary formula for ω^2

then $\omega^2 = \omega^2(A_1, A_2, \ldots A_n)$

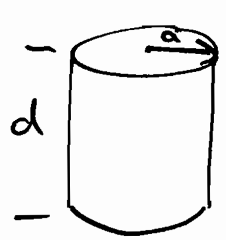
It can be shown that the best approximation to ω_r^2 is the one which minimizes $\omega^2(A_1, A_2, \ldots, A_n)$

$$\Rightarrow \quad \frac{\partial \omega^2}{\partial A_i} = 0 \qquad i = 1, 2, \ldots n$$

standard procedure is to set:

$$\bar{E}_{trial} = \sum_{n=1}^{N} A_n \bar{E}_n + \bar{E}_0$$

<u>Example</u> <u>circular cavity as before</u>



try $\bar{H} = \hat{a}_\phi (\rho + A\rho^2)$ $\quad \bar{\nabla} \times \bar{H} = \hat{a}_z (2 + 3A\rho)$

A- variational parameter.

a stationary Formula For $\bar{H}$ which requires no special B.C's is

$$\omega_r^2 = \frac{\iiint \epsilon^{-1} (\bar{\nabla} \times \bar{H})^2 \, dv}{\iiint \mu H^2 \, dv}$$

$$dv = \rho \, d\rho \, d\phi \, dz$$

$$\omega_r^2 = \frac{\int_0^d \int_0^{2\pi} \int_0^a \epsilon^{-1} \left[2 + 3A\rho \right]^2 \rho \, d\rho \, d\phi \, dz}{\mu \int_0^d \int_0^{2\pi} \int_0^a \left[\rho + A\rho^2 \right]^2 \rho \, d\rho \, d\phi \, dz}$$

$$= \frac{\int_0^a 4\rho + 12A\rho^2 + 9A^2\rho^3 \, d\rho}{\epsilon\mu \int_0^a \rho^3 + 2A\rho^4 + A^2\rho^5 \, d\rho}$$

$$= \frac{2a^2 + 4Aa^3 + \frac{9}{4}A^2a^4}{\epsilon\mu \left[\frac{a^4}{4} + \frac{2}{5}Aa^5 + \frac{A^2}{6}a^6 \right]}$$

$$= \frac{1}{a^2\epsilon\mu} \frac{8 + 16Aa + 9A^2a^2}{\left[1 + \frac{8}{5}Aa + \frac{2}{3}A^2a^2 \right]}$$

$$= \frac{15}{a^2\epsilon\mu} \frac{8 + 16Aa + 9(Aa)^2}{\left[15 + 24Aa + 10(Aa)^2 \right]}$$

now by the Rayleigh - Ritz method:

$$\frac{\partial \omega^2}{\partial A} = 0 = \left[16a + 18a^2 A\right]\left[15 + 24Aa + 10(Aa)^2\right] - \left[24a + 20a^2 A\right]\left[8 + 16Aa + 9(Aa)^2\right]$$

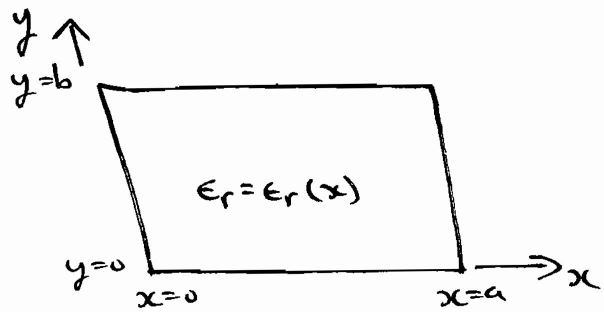
$$0 = 48a + 56a(Aa)^2 + 110a^2 A$$

$$0 = 24 + 55aA + 28(Aa)^2$$

roots: $Aa = \begin{cases} -1.3100 \\ -0.6543 \end{cases}$

$\therefore$ $\boxed{\omega = \dfrac{2.4087}{a\sqrt{\mu\epsilon}}}$ exact : $\omega_r = \dfrac{2.4048}{a\sqrt{\epsilon\mu}}$

Rayleigh - Ritz Method For the inhomogeneously Filled Rectangular Waveguide.



say that the dielectric constant varies with x-direction only.

if we define $\overline{\Pi}_e = \hat{a}_x \psi_e e^{-\gamma z}$ $(\gamma = k_z)$
(electric Hertz Potential)

then $\overline{H} = j\omega\epsilon_0 \overline{\nabla} \times \overline{\Pi}_e$ ① $\Rightarrow$ no H_x component

this produces what are called

Longitudinal Section Magnetic (LSM) modes
(TMx)

M.E.:

$$\begin{cases} \overline{\nabla} \times \overline{H} = j\omega\epsilon_0\epsilon_r(x)\,\overline{E} \\ \overline{\nabla} \times \overline{E} = -j\omega\mu_0\,\overline{H} \\ \overline{\nabla} \cdot \overline{B} = \mu\overline{\nabla} \cdot \overline{H} = 0 \\ \overline{\nabla} \cdot \overline{D} = 0 = \overline{\nabla} \cdot \epsilon_r \overline{E} \end{cases}$$

(automatically satisfied by $\overline{H} = j\omega\epsilon_0 \overline{\nabla} \times \overline{\Pi}_e$)

$$\therefore \quad \overline{\nabla} \times \overline{E} = k_0^2\, \overline{\nabla} \times \overline{\Pi}_e$$

$$\overline{E} = k_0^2\, \overline{\Pi}_e + \overline{\nabla}\phi$$

where ϕ is arbitrary scalar Fcn.

also: $\bar{\nabla} \times \bar{H} = j\omega\epsilon_0 \bar{\nabla}\times\bar{\nabla}\times\bar{\Pi}_e = j\omega\epsilon_0\epsilon_r(x)\,\bar{E}$

$$= j\omega\epsilon_0\epsilon_r(x)\left[k_0^2\,\bar{\Pi}_e + \bar{\nabla}\phi\right]$$

or $\bar{\nabla}\bar{\nabla}\cdot\bar{\Pi}_e - \bar{\nabla}^2\bar{\Pi}_e = \epsilon_r(x)\,k_0^2\,\bar{\Pi}_e + \epsilon_r(x)\bar{\nabla}\phi$

$$= \epsilon_r k_0^2 \bar{\Pi}_e + \bar{\nabla}(\phi\epsilon_r) - \phi\,\bar{\nabla}\epsilon_r$$

now letting:

$$\boxed{\bar{\nabla}\bar{\nabla}\cdot\bar{\Pi}_e = \bar{\nabla}(\epsilon_r\phi)} \qquad (2)$$

this was still arbitrary $\longrightarrow$ $\bar{\nabla}\cdot\bar{\Pi}_e = \epsilon_r\phi \Rightarrow \phi = \dfrac{\bar{\nabla}\cdot\bar{\Pi}_e}{\epsilon_r(x)} = \epsilon_r^{-1}\,\bar{\nabla}\cdot\bar{\Pi}_e$

$$\therefore \boxed{\bar{\nabla}^2\bar{\Pi}_e - \epsilon_r^{-1}(\bar{\nabla}\epsilon_r)\,\bar{\nabla}\cdot\bar{\Pi}_e + \epsilon_r k_0^2\,\bar{\Pi}_e = 0} \qquad (3)$$

and $\bar{E} = k_0^2\,\bar{\Pi}_e + \bar{\nabla}\left(\epsilon_r^{-1}\,\bar{\nabla}\cdot\bar{\Pi}_e\right)$

$$= k_0^2\,\bar{\Pi}_e + \bar{\nabla}\cdot\bar{\Pi}_e\;\bar{\nabla}\epsilon_r^{-1} + \epsilon_r^{-1}\,\bar{\nabla}\bar{\nabla}\cdot\bar{\Pi}_e$$

$$= k_0^2\bar{\Pi}_e - \epsilon_r^{-2}\,\bar{\nabla}\epsilon_r\,\bar{\nabla}\cdot\bar{\Pi}_e + \epsilon_r^{-1}\,\bar{\nabla}\bar{\nabla}\cdot\bar{\Pi}_e$$

$$= \epsilon_r^{-1}\left[\epsilon_r k_0^2\,\bar{\Pi}_e - \epsilon_r^{-1}\,\bar{\nabla}\epsilon_r\,\bar{\nabla}\cdot\bar{\Pi}_e + \bar{\nabla}\bar{\nabla}\cdot\bar{\Pi}_e\right]$$

$$\boxed{\bar{E} = \epsilon_r^{-1}\,\bar{\nabla}\times\bar{\nabla}\times\bar{\Pi}_e} \qquad (4)$$

and now $\bar{\nabla}\cdot\epsilon_r\bar{E} = 0$ is automatically satisfied.

Aside.

why not this way ??

if we would have let

$$\bar{\nabla}\bar{\nabla}\cdot\bar{\Pi}_e = \epsilon_r(x)\bar{\nabla}\phi$$

then $\boxed{\bar{\nabla}^2\bar{\Pi}_e + \epsilon_r k_0^2 \bar{\Pi}_e = 0}$

but then

$$\begin{aligned}\bar{E} &= k_0^2\,\bar{\Pi}_e + \bar{\nabla}\phi \\ &= k_0^2\bar{\Pi}_e + \epsilon_r^{-1}\bar{\nabla}\bar{\nabla}\cdot\bar{\Pi}_e \\ &= \epsilon_r^{-1}\left[-\bar{\nabla}^2\bar{\Pi}_e + \bar{\nabla}\bar{\nabla}\cdot\bar{\Pi}_e\right] \\ &= \epsilon_r^{-1}\left[\bar{\nabla}\times\bar{\nabla}\times\bar{\Pi}_e\right]\end{aligned}$$

and again $\bar{\nabla}\cdot\epsilon_r\bar{E} = 0$ is automatically satisfied.

since there is no change in ϵ_r w.r.t the y direction, we assume

$$\bar{\Pi}_e = \hat{a}_x \, \Phi_e(x) \sin \frac{m\pi y}{b} \, e^{-\gamma z}$$

$$\bar{\nabla}^2 \bar{\Pi}_e = \hat{a}_x \frac{\partial^2}{\partial x^2} \Phi_e(x) \left(\sin \frac{m\pi y}{b} \, e^{-\gamma z} \right) + \left(\left(\frac{m\pi}{b}\right)^2 + \gamma^2 \right) \Phi_e \sin \frac{m\pi y}{b} e^{-\gamma z}$$

$$\epsilon_r^{-1} \bar{\nabla} \epsilon_r (\bar{\nabla} \cdot \bar{\Pi}_e) = \hat{a}_x \frac{\partial \epsilon_r}{\partial x} \frac{\partial \Phi_e(x)}{\partial x} \left(\sin \frac{m\pi y}{b} \, e^{-\gamma z} \right) \epsilon_r^{-1}$$

$$\epsilon_r k_o^2 \bar{\Pi}_e = \hat{a}_x \epsilon_r k_o^2 \, \Phi_e(x) \sin \frac{m\pi y}{b} \, e^{-\gamma z}.$$

$$\therefore \quad \frac{d^2 \Phi_e}{dx^2} - \epsilon_r^{-1} \frac{d\epsilon_r}{dx} \frac{d\Phi_e}{dx} + (\epsilon_r k_o^2 + \gamma^2 - h^2) \, \Phi_e(x) = 0$$

or (ODE)

$$\boxed{\frac{d}{dx}\left[\frac{1}{\epsilon_r} \frac{d\Phi_e}{dx}\right] + \frac{1}{\epsilon_r} (\epsilon_r k_o^2 + \gamma^2 - h^2) \, \Phi_e = 0} \quad (5)$$

$$h = \frac{m\pi}{b}$$

$$\bar{E} = \epsilon_r^{-1} \, \bar{\nabla} \times \bar{\nabla} \times \bar{\Pi}_e = \epsilon_r^{-1} \left[\hat{a}_x \left(\frac{\partial^2}{\partial z^2} \Pi_x + \frac{\partial^2}{\partial y^2} \Pi_x \right) + \hat{a}_y \left(\frac{-\partial^2}{\partial x \partial y} \Pi_x \right) + \hat{a}_z \left(\frac{\partial^2}{\partial x \partial z} \Pi_x \right) \right]$$

since $E_y = E_z = 0$ at $x = 0, a$

the appropriate B.C. on $\Phi_e(x)$ is

B.C. (Neumann) $$\boxed{\frac{d\Phi_e}{dx} = 0 \qquad x = 0, a} \qquad (6)$$

eq's (5) & (6) form a Sturm-Liouville system

having a complete, orthonormal set of eigenfunctions

$$\phi_{en}$$

such that $$\boxed{\int_0^a \epsilon_r^{-1} \phi_{en} \phi_{em} \, dx = \delta_{nm}}$$ orthonormal

Note the term $r(x) = \epsilon_r^{-1}$ in the integral.

an <u>arbitrary</u> fcn $g(x)$ which is piecewise-continuous may be expanded as

$$\begin{cases} g(x) = \sum_{n=0}^{\infty} a_n \phi_{en} \\ a_n = \int_0^a \epsilon_r^{-1} g(x) \phi_{en} \, dx \end{cases}$$

If we multiply the ODE by $\bar{\Phi}_e$ and integrate from 0 to a

$$\gamma^2 \int_0^a \Phi_e^2 \, \epsilon_r^{-1} \, dx = \int_0^a \bar{\Phi}_e \left[-\frac{d}{dx}\left(\epsilon_r^{-1} \frac{d\bar{\Phi}_e}{dx} \right) - \left(k_0^2 - \frac{h^2}{\epsilon_r} \right) \bar{\Phi}_e \right] dx$$

$$= \int_0^a \epsilon_r^{-1} \left[\left(\frac{d\bar{\Phi}_e}{dx} \right)^2 - \left(\epsilon_r k_0^2 - h^2 \right) \bar{\Phi}_e^2 \right] dx$$

(by integration by parts),

we have an expression for the propagation constant γ^2. and it can be shown that this expression is stationary by ($\delta\gamma^2$??) taking the first variation $\delta\Phi_e$

now consider the function Φ which is expanded in terms of eigenfcns and is orthonormal to the first K true eigenfunctions

then: $\Phi = \sum_{n=0}^{\infty} a_n \phi_{en}$

and $a_n = \int_0^a \epsilon_r^{-1} \Phi \, \phi_{en} \, dx = 0 \qquad n = 0, 1, \ldots, K-1$

if we substitute Φ into our variational formula then it can be shown that

$$\gamma^2 \sum_{n=k}^{\infty} a_n^2 = \sum_{n=k}^{\infty} a_n^2 \gamma_n^2$$

γ_n are eigenvalues of equation

$$\frac{d}{dx}\left[\frac{1}{\epsilon_r}\frac{d\Phi_e}{dx}\right] + \epsilon_r^{-1}\left(\epsilon_r k_0^2 + \gamma_n^2 - h^2\right)\Phi_e = 0$$

now assuming $\gamma_0^2 < \gamma_1^2 < \gamma_2^2 \ldots < \gamma_k^2$

$$\boxed{\gamma^2 = \frac{\sum_{n=k}^{\infty} a_n^2 \gamma_n^2}{\sum_{n=k}^{\infty} a_n^2} = \gamma_k^2 + \frac{\sum_{n=k+1}^{\infty} a_n^2 (\gamma_n^2 - \gamma_k^2)}{\sum_{n=k}^{\infty} a_n^2} \geq \gamma_k^2}$$

$\therefore$ γ^2 is an upper bound to γ_k^2

$\therefore$ if Φ is not orthogonal to any eigen fcns then
γ^2 is an upper bound to γ_0

example: we choose the empty guide eigenfcns:

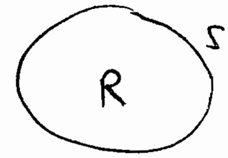
$$\begin{cases} f_n(x) = \left(\dfrac{\epsilon_{0n}}{a}\right)^{\frac{1}{2}} \cos \dfrac{n\pi x}{a} & n = 0, 1, \ldots \\ \epsilon_{0n} = \begin{cases} 1 & n = 0 \\ 2 & n > 0 \end{cases} & \text{Neumann No.} \end{cases}$$

this will not be orthogonal to any of the real eigenfunctions (in general) and thus γ will be an upper bound to the fundamental mode.

General Eigenvalues:

consider:

$$\nabla^2 \psi + \lambda \psi = 0 \qquad (1)$$



in a 3-D region with

$$\frac{d\psi}{dn} + \alpha \psi = 0 \quad \text{on} \quad S \qquad (2)$$

define:

$$\frac{Q(F)}{N(F)} = \rho(F) = \frac{\iiint\limits_R (\nabla F \cdot \nabla F)\, dV + \iint\limits_S \alpha F^2\, dS}{\iiint\limits_R F^2\, dV} \qquad (3)$$

where F are taken from the class of functions which are continuous and have piecewise continuous first derivatives in R.

1) The minimum of $\rho(F)$ will equal the lowest eigenvalue of (1) and the minimizing fcn will be the first eigenfcn.

Next minimize $\rho(F)$ subject to

$$\iiint\limits_R F \psi_1\, dV = 0$$

this next minimum will be λ_2 and the minimizing fcn will be ψ_2.

k^{th} eigenvalue is obtained by minimizing $\rho(F)$ subject to

$$\iiint_R F \psi_j \, dV = 0 \qquad j = 1, 2, \ldots k-1.$$

The eigenvalues, and eigen fcns so derived are orthogonal but not necesarily orthonormal.